Logs and the Exponential e

Q What is a logarithm?   A The inverse of an exponential

for example, taking the general case;

if $y = a^x$ where a is a positive constant but not 1, then by definition

$\log_a y = x$ (log to the base a of y equals x).

An early use of logs was to turn multiplication into addition to facilitate astronomical table compilation – this will be clearer when the 'log' rules are defined.

Basic log rules

for manipulation of logs – apply to any common base

1. $\log y + \log z = \log (yz)$
2. $\log y - \log z = \log (y/z)$
3. $(\log y^n) = n \log y$

Proof

1. For base a, if $y = a^x$, $z = a^w$ then $yz = a^{x+w}$ (law of indices),

   then by definition $\log(yz) = x + w = \log y + \log z$

2. Similar to 1.

3. For base a, if $y = a^x$, then $y^n = (a^x)^n = a^{xn}$ so $(\log y^n) = xn = n \log y$

Change of base.

Logs to base b can be changed to logs to base a using

$\log_a y = \log_b y/\log_b a$

Proof

if $y = a^x$, taking logs to base b, $\log_b y = \log_b(a^x) = x \log_b(a) = \log_a y \log_b(a)$

hence $\log_a y = \log_b y/\log_b a$

Also $\log_a b \log_b a = 1$

Differential of logs

Let y = log x
then by definition of differentiation,

$$d\frac{(\log x)}{dx} = _{limit\,\delta x \to 0} \frac{\log(x+\delta x) - \log(x)}{\delta x}$$

$$........ = \frac{\log((x+\delta x)/x)}{\delta x} = \frac{x}{x}\frac{1}{\delta x}\log(1+\frac{1}{(x/\delta x)})$$

$$writing\ n = x/\delta x$$

$$.... = \frac{1}{x}\log(1+\frac{1}{n})^n$$

$$now\ one\ definition\ of\ e\ is\ ....\ e = _{(limit\ n\to\infty)}(1+\frac{1}{n})^n$$

$$remembering\ that\ \log_a a = 1$$

$$gives\ d\frac{(\log x)}{dx} = \frac{1}{x}\ if\ the\ \log\ base\ is\ e$$

$$for\ base\ a\ the\ result\ gives\ d\frac{(\log_a x)}{dx} = \frac{1}{x}\log_a e$$

Euler's Relation

e^{ix} = cos x + isin x

proof – (can also be proved using series expansion)

$$let\ y = \cos x + i\sin x$$

$$\frac{dy}{dx} = -\sin x + i\cos x = i(\cos x + i\sin x) = iy$$

$$\int\frac{1}{y}dy = ix \to \to \ln|y| = i\theta + c$$

$$when\ x=0,\ y=1 \to c=0$$

$$\to y = e^{ix}$$

$$\to e^{ix} = \cos x + i\sin x$$

More about 'e'

$$\frac{de^x}{dx} = e^x$$

$$Proof: let\ y\ be\ a\ function\ of\ x\ such\ that\ \frac{dy}{dx} = y$$

$$then \int \frac{1}{y}\, dx = \int dx$$

$$\rightarrow x = \ln|y| + c$$

$$\rightarrow y = e^{x-c} = e^x e^{-c} = ke^x\ for\ any\ k$$

$$\rightarrow \text{and}\ so\ when\ k = 1,$$

$$\frac{de^x}{dx} = e^x$$

Numerical value of e = 2.718281828459.....

e is an irrational number and hence is neither a recurring decimal, nor a fraction.

Remembering that π is also irrational and = 3.141592...

a suprising result from Euler's relation is

$$e^{i\pi} = -1$$

that is mixing 2 irrational numbers with an imaginary number results in -1. How amazing is that?